Fractional Order Ubriaco Entropy and its Generalizations: New Paradigms for Decision-Making, Flow Dynamics and Survival Analysis

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Abstract

Information theory provides a foundational framework for quantifying uncertainty, complexity, and disorder in diverse natural and engineered systems. Its origins trace back to Shannon's pioneering work, where the entropy concept was first introduced to measure uncertainty in communication signals, enabling the theoretical demonstration of error-free information transmission through noisy channels. Since then, Shannon's entropy has become a cornerstone of modern modeling and analysis, finding applications in fields as varied as financial decision theory, hydraulic modeling, and biomedical survival analysis. However, modern complex systems and many real world phenomena exhibit dynamic features, such as nonlinear interactions, long-memory effects, anomalous transport, structural heterogeneity, and heavy-tailed fluctuations, that classical entropy-based models fail to capture adequately. Traditional approaches often assume local interactions, short-term dependencies, or linear variability, limiting their ability to describe systems governed by long-range correlations, turbulence, stochastic volatility, or complex lifetime dynamics. In parallel, the growing field of fractional calculus has introduced nonlocal and memory-dependent operators that more accurately describe anomalous transport, long-range interactions, and fractional order behaviors. The intersection of these two domains has given rise to fractional information measures, particularly fractional order entropy, which offer enhanced flexibility and descriptive richness by tuning the sensitivity of the entropy measure through a fractional parameter. Although fractional calculus has emerged as a promising mathematical tool for representing long-range memory and nonlocality, its application within entropy-based frameworks remains limited. This gap highlights a longstanding challenge: the need to explore both the theoretical foundations and practical utility of fractional order information measures, particularly the fractional entropy measure proposed by Ubriaco (2009), in characterizing the complex behavior of physical processes and modern systems.

This study of fractional order Ubriaco entropy and its generalizations develops a unified information-theoretic framework that advances both the theoretical foundations and practical applications of entropy across multidisciplinary domains. In the realm of financial decision theory, fractional order Ubriaco entropy is employed to formulate two new risk

measures, enabling the development of risk-sensitive decision-making models that support optimized portfolio selection while accommodating diverse investor attitudes or risk tolerance levels, ranging from risk-averse (conservative) to risk-seeking (adventurous). This flexibility is achieved by tuning the fractional parameter from 0 (conservative) to 1 (adventurous). Within environmental hydraulics, the continuous form of fractional Ubriaco entropy termed as fractional differential entropy (FDE), provides an improved basis for modeling turbulent flow vertical velocity profiles and suspended sediment transport dynamics along the vertical section in open-channel systems. In biomedical survival analysis, newly developed quantile-based fractional generalized cumulative past entropy measure capture complex lifetime uncertainty patterns and censoring mechanisms that classical survival metrics fail to detect. Thus, this work demonstrates the versatility and unifying strength of fractional order entropy measures in addressing challenges across finance, hydrodynamics and medical survival studies through six major research contributions spanning Chapters 2–7, as summarized below.

Chapter 1 introduces the principal concepts and provides detailed explainatory notes on the ideas used in the study.

Chapter 2 introduces a novel decision-making framework for stock selection based on fractional-order entropy due to Ubriaco. By tuning the fractional parameter α, the model captures varying attitudes of individuals toward risk. Values of α near 1 indicate high risk tolerance (adventurous attitude), while those near 0 reflect risk aversion (conservative attitude). The sensitivity of the fractional order entropy to changing risk preferences of decision-makers is demonstrated through four real-world portfolio models, namely, large-cap, mid-cap, diversified, and hypothetical. Furthermore, two new risk measures, termed as expected utility–fractional entropy (EU-FE) and expected utility-fractional entropy and variance (EU-FEV), are introduced to develop decision models aligned with investors' risk preferences. The performance of the developed models are compared with existing decision-making approaches, thereby demonstrating its superiority and consistency. The effectiveness of the decision model is further tested with financial stock market data of Portuguese stock market (PSI 20) by finding efficient frontiers of portfolio with the aid of *Artificial Neural Network* (ANN).

The fractional order entropy due to Ubriaco was introduced as an uncertainty measure to capture investors' varying attitudes toward risk in Chapter 2. Inspired by these developments, we introduce the notion of normalized fractional order entropy aligned with investors' risk preferences in **Chapter 3**. Hereafter, we propose two novel risk measures, termed normalized expected utility–fractional entropy (NEU-FE) and normalized expected utility–fractional entropy–variance (NEU-FEV), which combine normalized fractional entropy with expected utility and variance. The risk sensitivity is controlled by the fractional parameter q, interpolating between conservative or risk aversion ($q \to 0$) and adventurous or high-risk tolerance ($q \to 1$) attitudes. The proposed risk measures form the basis of two decision making frameworks applied to the Indian stock market data (NIFTY 50) to identify an optimal low-risk set of five stocks from a broader list. Comparative analyses with existing fractional entropy-based models, supported by machine learning techniques (Random Forest, Ridge Regression, Lasso Regression, and artificial neural networks), demonstrate the robustness of the proposed measures. Statistical significance tests further validate model reliability and identify the most effective approach for ranking stocks in accordance with investor preferences.

In the preceding chapters, we work on Ubriaco entropy for discrete random variables. In **Chapter 4**, we conduct a detailed study of the continuous analogue of this entropy termed as fractional differential entropy (FDE) and find some interesting properties which makes it stand out among the existing entropies in literature. The studied entropy measure is evaluated analytically and numerically for some well-known continuous distributions, which will be quite useful in reliability analysis works and other statistical studies of complex systems. Further, it has been used to model the one-dimensional vertical velocity profile of turbulent flows in wide open channels. A one-parametric spatial distribution function is utilized for better estimation of the velocity distribution. The validity of the model has been established using experimental and field data through regression analysis. A comparative study is also presented to show the superiority of the proposed model over the existing entropy-based hydraulic models.

Accurately characterizing the vertical distribution of suspended sediment concentration (SSC) in open-channel flows is essential for reliable riverine sediment management and modeling. **Chapter 5** introduces a simplified yet robust model for SSC estimation, grounded in fractional differential entropy (FDE) theory. The normalized SSC is modeled as a continuous random variable, subject to a physically realistic boundary condition of zero concentration at the water surface. Using a leading-order approximation approach, the problem is then expressed in terms of a power series solution within a maximum entropy optimization framework. The resulting closed-form solution for the SSC profile is rigorously validated against both experimental data and real-world field measurements. Comparative analyses demonstrate that the proposed FDE-based model not only achieves higher predictive accuracy but also offers substantial computational efficiency relative to conventional deterministic and classical entropy-based methods. Further the sensitivity of the parameters of the proposed model is investigated through resampling methods such as bootstrap and Jackknife empirical likelihood methods. These findings suggest that the FDE approach provides an effective and practical tool for sediment transport modeling in natural and engineered waterways.

Open channel flow often carries toxic wastes and non-degradable materials as sediments, which negatively impact marine life and water quality. Therefore, monitoring the distribution of suspended sediment concentration (SSC) in open channel flows is of critical importance for effective water resource management. In **Chapter 6**, the time-averaged normalized sediment concentration is treated as a random variable to derive its optimal probability density function (pdf) using the fractional order entropy proposed by Machado. The Lambert W function is employed to derive the pdf under a specified convergence criterion, which is then used to obtain the vertical concentration distribution. To address the computational complexity of determining the Lagrange multipliers and the entropy index in the fractional entropy-based

concentration model, a new optimization (minimization) problem is formulated that incorporates the convergence condition. The proposed model is validated using both experimental and field data sets. Additionally, regression and error analyses are conducted to demonstrate the model's advantages over extant entropy-based probabilistic and classical deterministic models.

Uncertainty in past lifetime distributions and the timing of inactivity in systems and their components has been effectively quantified using the recently introduced concept of fractional generalized cumulative past entropy (FGCPE) which is motivated by the generalized entropy framework due to Machado for fractional orders greater than zero. Building on these concepts, we propose a quantile-based variant, the quantile fractional generalized cumulative past entropy (QFGCPE), along with its dynamic time-dependent counterpart (DQFGCPE) in **Chapter 7**. This new measure of lifetime uncertainty are used as tools for uncertainty quantification in lifetime and survival analysis. We derive closed-form expressions for several lifetime distributions, thereby showing that by integrating quantile representations with fractional weighting, the measure remains robust without relying on density assumptions. Hereafter, we derive key analytical properties such as bounds, monotonicity, and stochastic orderings. Next, we develop nonparametric and Kaplan-Meier-based estimators for complete and right-censored data and validated through simulation studies. Additional analysis using the logistic map highlights the sensitivity of the measure to transitions between regular and chaotic regimes. Further, by applying to the NCCTG lung cancer dataset, QFGCPE reveals clinically meaningful differences in early-failure uncertainty between patient subgroups-patterns that classical survival summaries fail to detect.

In summary, this thesis demonstrates that fractional-order Ubriaco entropy and its generalizations due to Machado present a versatile and powerful extension of classical information theory. Through new formulations, modeling frameworks, and extensive empirical validation across finance, hydrodynamics and biomedical survival analysis, the work establishes fractional entropy as a unifying paradigm for uncertainty quantification and complex system modeling.

Key Words and Phrases: Flow velocity, fractional generalized cumulative past entropy, fractional order entropy, Machado entropy, risk measures, suspended sediment concentration, Ubriaco entropy.